

The lattice size of a lattice polygon

Wouter Castryck* and Filip Cools

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Abstract

We give an upper bound on the minimal degree of a plane model of a given plane algebraic curve, in terms of the combinatorics of the Newton polygon of its defining polynomial. We prove in various cases that this bound is sharp as soon as the polynomial is sufficiently generic with respect to its Newton polygon. The central combinatorial notion here is the *lattice size* of a lattice polygon, which can be seen as a two-dimensional analogue of the more commonly studied lattice width. Our main auxiliary tool is an expression for the lattice size of a lattice polygon in terms of the lattice size of its interior, thereby providing a recursive method for computing the lattice size in practical situations.

MSC2010: Primary 14H45, Secondary 14H51, 14M25

1 Introduction

Let k be an algebraically closed field, which we assume¹ to be of characteristic 0. Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be an irreducible Laurent polynomial, whose Newton polygon we denote by $\Delta(f)$. Finally, let $\mathbb{T}^2 = k^* \times k^*$ be the two-dimensional torus over k , and let $U_f \subset \mathbb{T}^2$ be the curve cut out by f . The aim of this note is to give an upper bound on $s_2(U_f)$, the minimal degree of a (possibly singular) plane projective curve that is birationally equivalent to U_f , purely in terms of the combinatorics of $\Delta(f)$.

In general, for an algebraic curve C/k , the invariant $s_2(C)$ is not well-understood, although it is known that

$$\frac{3 + \sqrt{8g + 1}}{2} \leq s_2(C) \leq g + 2,$$

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¹This assumption is made because some of our references make it as well. Our main result Theorem 1 does not depend on these references and is valid in any characteristic.

where g is the geometric genus of C . The lower bound is met if and only if C is birationally equivalent to a non-singular projective plane curve, while the upper bound is met if and only if C is elliptic or hyperelliptic. See [5] and the references therein for proofs and some more facts.

Our central combinatorial notion is the *lattice size* of a lattice polygon Δ , denoted $\text{ls}(\Delta)$. In case $\Delta \neq \emptyset$ we define it as the smallest integer $d \geq 0$ for which there exists a unimodular transformation $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$\varphi(\Delta) \subset d\Sigma.$$

Here $\Sigma = \text{conv}\{(0, 0), (1, 0), (0, 1)\}$ is the standard simplex, and the multiple $d\Sigma$ is in Minkowski's sense.



It is convenient to define $\text{ls}(\emptyset) = -2$.

Remark. If in this definition one replaces $d\Sigma$ by $\mathbb{R} \times [0, d]$, one gets the more classical *lattice width* of Δ , denoted $\text{lw}(\Delta)$. (Here it is more natural to define $\text{lw}(\emptyset) = -1$.)

In Section 4 we will prove Theorem 6, which expresses $\text{ls}(\Delta)$ in terms of $\text{ls}(\Delta^{(1)})$, where $\Delta^{(1)}$ denotes the convex hull of the lattice points in the interior of Δ . The proof is purely combinatorial and forms the bulk of our paper. This provides a recursive method for computing $\text{ls}(\Delta)$ in practical situations, by gradually ‘peeling off’ the polygon.

Since unimodular transformations induce automorphisms of \mathbb{T}^2 , it is immediate that $s_2(U_f) \leq \text{ls}(\Delta(f))$. Our main result, which shows up as a consequence to Theorem 6, gives an upper bound for $s_2(U_f)$ in terms of $\text{ls}(\Delta(f)^{(1)})$. In order to state the result, we define $\Upsilon = \text{conv}\{(-1, -1), (1, 0), (0, 1)\}$, and when we say that two lattice polygons Δ and Δ' are *equivalent* (notation: $\Delta \cong \Delta'$), we mean that Δ' is obtained from Δ through a unimodular transformation.

Theorem 1. *One has*

$$s_2(U_f) \leq \text{ls}(\Delta(f)^{(1)}) + 3. \tag{1}$$

If $\Delta(f) \cong d\Upsilon$ for a certain integer $d \geq 2$, then moreover $s_2(U_f) \leq 3d - 1$.

We remark that $\text{ls}(\Delta^{(1)}) + 3 \leq \text{ls}(\Delta)$ (see (2) below), and that the difference can be arbitrarily large; see Section 3 for examples. Hence Theorem 1 can be seen as a considerable sharpening of the trivial bound $s_2(U_f) \leq \text{ls}(\Delta(f))$. In fact, we conjecture that if f is sufficiently generic with respect to its Newton polygon, then

the (smallest applicable) bound of Theorem 1 is met. In Section 5, where we will be more precise on what we mean by ‘sufficiently generic’, we will prove this conjecture in a number of special cases.

2 First properties of the lattice size

Let $d \in \mathbb{Z}_{\geq 0}$. Then $\text{ls}(\text{conv}\{(0,0), (d,0)\}) = d$: indeed, it is immediate that $\text{conv}\{(0,0), (d,0)\} \subset d\Sigma$ and that the integral distance $\gcd(a_2 - a_1, b_2 - b_1)$ between two points $(a_1, b_1), (a_2, b_2) \in (d-1)\Sigma$ cannot exceed $d-1$. More generally, every lattice polygon that contains a line segment of integral length d must have lattice size at least d . In particular $\text{ls}(d\Sigma) = d$.

Lemma 2. *Let Δ be a lattice polygon. Then $\text{lw}(\Delta) \leq \text{ls}(\Delta)$, and equality holds if and only if $\Delta \cong d\Sigma$ for some integer $d \geq 0$.*

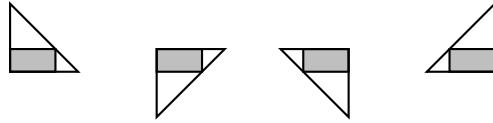
PROOF. This follows because $\text{lw}(d\Sigma) = d$, while every strict subpolygon $\Gamma \subset d\Sigma$ satisfies $\text{lw}(\Gamma) < d$. ■

A less straightforward lattice size calculation is:

Lemma 3. *Let $a, b \in \mathbb{Z}_{\geq 0}$ and consider $\square_{a,b} = \text{conv}\{(0,0), (a,0), (a,b), (0,b)\}$. Then $\text{ls}(\square_{a,b}) = a + b$.*

PROOF. The case where $a = 0$ or $b = 0$ follows from the above considerations, so we can assume that $a, b \geq 1$. Instead of looking for the minimal d such that $\square_{a,b}$ can be mapped inside $d\Sigma$ through a unimodular transformation, we will look for the minimal d such that $\square_{a,b}$ is contained in a unimodular transform of $d\Sigma$. More precisely, we will prove the following assertion by induction on $a + b$:

We have $\text{ls}(\square_{a,b}) = a + b$. Moreover, there are exactly four ways of fitting $\square_{a,b}$ inside a unimodular transform of $(a + b)\Sigma$:



The basis of our induction is the case $a = b = 1$. Here, the first part of the assertion holds because $\square_{1,1} \subset 2\Sigma$ and $\text{Vol}(\square_{1,1}) > \text{Vol}(\Sigma)$. The second part follows because 2Σ contains only 3 lattice points that are non-vertices. Therefore, when fitting $\square_{1,1}$ inside a transform of 2Σ , at least one of its vertices must coincide with a vertex of 2Σ , and the two adjacent vertices of $\square_{1,1}$ must coincide with the interior lattice points of the respective adjacent edges of 2Σ . From this the claim follows easily.

Now assume that $a, b \geq 1$ and (without loss of generality) that $a \geq 2$. Clearly $\square_{a,b} \subset (a+b)\Sigma$. Suppose that $\square_{a,b}$ sits inside a unimodular transform of $(a+b-1)\Sigma$. By applying the induction hypothesis to $\square_{a-1,b} \subset \square_{a,b}$ we find that $(a+b-1)\Sigma$ must enclose this subpolygon in one of the four manners above. But for each of these four configurations, it is clear that $\square_{a,b}$ itself could not have been contained in $(a+b-1)\Sigma$: contradiction. As for the second assertion, let Σ' be a unimodular transform of $(a+b)\Sigma$ containing $\square_{a,b}$. Then

- each edge of Σ' must contain at least one vertex of $\square_{a,b}$: otherwise we could crop Σ' to a unimodular transform of $(a+b-1)\Sigma$ that still contains $\square_{a,b}$;
- at least one vertex v of Σ' does not appear as a vertex of $\square_{a,b}$: otherwise the latter would be a triangle;
- the edges of Σ' that are adjacent to v cannot contain two vertices of $\square_{a,b}$ each: otherwise $\square_{a,b}$ would contain two non-adjacent non-parallel edges.

So there must be an edge $\tau \subset \Sigma'$ that contains exactly one vertex v of $\square_{a,b}$. Then the transform of $(a+b-1)\Sigma$ obtained from Σ' by shifting τ inwards contains $\square_{a,b} \setminus \{v\}$. In particular it contains (a translate of) $\square_{a-1,b}$. By applying the induction hypothesis we find that Σ' must be positioned in one of the four standard ways above. ■

3 A recursive formula for computing $\text{ls}(\Delta)$

We now investigate the relation between $\text{ls}(\Delta)$ and $\text{ls}(\Delta^{(1)})$. Since for $d \geq 3$ one has $(d\Sigma)^{(1)} = (d-3)\Sigma$, we have that

$$\text{ls}(\Delta^{(1)}) \leq \text{ls}(\Delta) - 3 \quad (2)$$

as soon as Δ is two-dimensional (this includes the case where $\Delta^{(1)} = \emptyset$, which can be verified separately). Typically, one expects equality to hold, but there are many exceptions, which are classified by Theorem 6 below.

In what follows, we will make use of the following terminology and facts; see [3, §4] or [6, §2.2] for proofs. An edge τ of a two-dimensional lattice polygon Γ is always supported on a line $a_\tau X + b_\tau Y = c_\tau$ with $a_\tau, b_\tau, c_\tau \in \mathbb{Z}$ and a_τ, b_τ coprime. When signs are chose appropriately, we can moreover assume that Γ is contained in the half-plane $a_\tau X + b_\tau Y \leq c_\tau$. The line $a_\tau X + b_\tau Y = c_\tau + 1$ is called the *outward shift* of τ . It is denoted by $\tau^{(-1)}$, and the polygon (which may take vertices outside \mathbb{Z}^2) that arises as the intersection of the half-planes $a_\tau X + b_\tau Y \leq c_\tau + 1$ is denoted by $\Gamma^{(-1)}$. If $\Gamma = \Delta^{(1)}$ for some lattice polygon Δ , then the outward shifts of two adjacent edges of Γ always intersect in a lattice point, and in fact $\Gamma^{(-1)} = \Delta^{(1)(-1)}$ is a lattice polygon. Moreover, $\Delta \subset \Delta^{(1)(-1)}$, i.e. $\Delta^{(1)(-1)}$ is the maximal lattice

polygon (with respect to inclusion) for which the convex hull of the interior lattice points equals $\Delta^{(1)}$.

Before stating Theorem 6, let us prove two auxiliary lemmas:

Lemma 4. *Assume that there exist parallel edges $\tau \subset \Delta$ and $\tau' \subset \Delta^{(1)}$ whose supporting lines are at integral distance 1 of each other, of respective lengths r and s . If $r \geq s + 3$ then $\text{ls}(\Delta^{(1)}) = s$ and $\text{ls}(\Delta) = r$.*

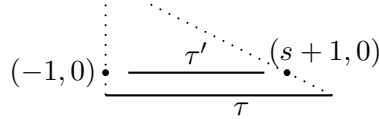
Remark. As ever, by an edge we mean a one-dimensional face. In particular, if $\Delta^{(1)}$ is one-dimensional then it is an edge of itself. Example: consider the hyperelliptic Weierstrass polygon

$$\text{conv}\{(0, 0), (2g + 1, 0), (0, 2)\}.$$

Then $\text{ls}(\Delta^{(1)}) = g$ and $\text{ls}(\Delta) = 2g + 1$. This shows that the difference between $\text{ls}(\Delta)$ and $\text{ls}(\Delta^{(1)})$ can be arbitrarily large.

PROOF OF LEMMA 4. By using a unimodular transformation if needed, we can assume that $\tau = \text{conv}\{(-1, -1), (r - 1, -1)\}$ and $\tau' = \text{conv}\{(0, 0), (s, 0)\}$. Since $r \geq s + 3$ and $\Delta^{(1)}$ cannot contain any lattice points on the line $Y = 0$ apart from those contained in τ' ,

- the edge of Δ that is left-adjacent to τ must pass through or to the right of $(-1, 0)$, and
- the edge of Δ that is right-adjacent to τ must pass through or to the left of $(s + 1, 0)$.



From the convexity of Δ one immediately sees that $\Delta \subset (-1, -1) + r\Sigma$, and similarly that $\Delta^{(1)} \subset s\Sigma$. Therefore $\text{ls}(\Delta^{(1)}) \leq s$ and $\text{ls}(\Delta) \leq r$, and equality follows from the considerations preceding Lemma 3. ■

Lemma 5. *Assume that $\Delta^{(1)}$ is two-dimensional. Let $s \geq 1$ be an integer such that $\Delta^{(1)} \subset s\Sigma$, in such a way that $\Delta^{(1)}$ has an edge τ' in common with $s\Sigma$. Let $\tau'^{(-1)}$ be its outward shift, and consider the face $\tau = \Delta \cap \tau'^{(-1)}$ of Δ , whose integral length we denote by r . Then*

$$\text{ls}(\Delta^{(1)}) = s \quad \text{and} \quad \text{ls}(\Delta) = \max\{r, s + 3\}.$$

Remark. The face τ is either a vertex or an edge. In the former case, its integral length is understood to be 0.

PROOF. The fact that $\text{ls}(\Delta^{(1)}) = s$ follows immediately from the considerations preceding Lemma 3. As for $\text{ls}(\Delta)$, in case $r \geq s + 3$ the statement follows from Lemma 4. So assume that $r \leq s + 3$ (we reinclude the case $r = s + 3$ for the sake of the symmetry of the argument below). Without loss of generality we may suppose that $\tau' = \text{conv}\{(0, 0), (s, 0)\}$. We claim that we can moreover assume that $\tau \subset \text{conv}\{(-1, -1), (s + 2, -1)\}$, while still keeping $\Delta^{(1)} \subset s\Sigma$.

Assuming the claim, we can make the following reasoning.

- Clearly Δ is contained in the half-plane $Y \geq -1$.
- Suppose that Δ contains a lattice point (a, b) for which $a < -1$. Because $b = -1$ contradicts our claim, while $b = 0$ contradicts that $\Delta^{(1)} \subset s\Sigma$ (indeed, it implies that $(-1, 0) \in \Delta^{(1)}$), we must have $b \geq 1$. Along with the fact that $\Delta^{(1)}$ is two-dimensional (so that it must contain a lattice point on or above the line $Y = 1$) this implies that $(0, 1) \in \Delta^{(1)}$. But then, apart from the point (a, b) itself, all lattice points which are contained in the triangle spanned by (a, b) , $(0, 0)$ and $(0, 1)$ must be elements of $\Delta^{(1)}$. The volume of this triangle being at least 1, Pick's theorem implies that it must contain a lattice point different from (a, b) , $(0, 0)$ and $(0, 1)$. This contradicts $\Delta^{(1)} \subset s\Sigma$.

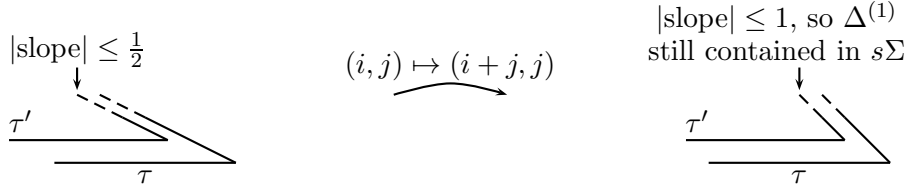
We conclude that Δ is contained in the half-plane $X \geq -1$.

- By applying the unimodular transformation $(i, j) \mapsto (s - i - j, j)$, one sees that the foregoing reasoning also allows to conclude that Δ is contained in the half-plane $X + Y \leq s + 1$.

So the claim implies that $\Delta \subset (-1, -1) + (s + 3)\Sigma$, and hence that $\text{ls}(\Delta) \leq s + 3$, which together with (2) proves the lemma.

To prove the claim, note that because $r \leq s + 3$, again using the transformation $(i, j) \mapsto (s - i - j, j)$ if needed, we can assume that τ is contained in the half-plane $X \geq -1$. Let $(a, -1)$ be the right-most vertex of τ . As long as $a > s + 2$, we can apply a unimodular transformation of the form $(i, j) \mapsto (i + j, j)$ to Δ , while

- keeping τ in the half-plane $X \geq -1$ (here we again used that $r \leq s + 3$);
- keeping $\Delta^{(1)}$ inside $s\Sigma$: indeed, because $a > s + 2$ and $(s + 1, 0) \notin \Delta^{(1)}$, the edge of Δ that is right-adjacent to τ must have a slope that is smaller than $1/2$ (in absolute value), and hence the same must be true for the edge of $\Delta^{(1)}$ that is right-adjacent to τ' .

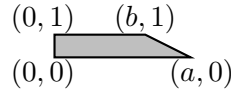


This decreases the value of a by 1. So the claim follows by repeating this step until $a \leq s + 2$. \blacksquare

We are now ready to state and prove Theorem 6.

Theorem 6. *Let Δ be a two-dimensional lattice polygon. Then $\text{ls}(\Delta^{(1)}) = \text{ls}(\Delta) - 3$, except in the following situations:*

- Δ is equivalent to a Lawrence prism



where $a = b = 1$ or $2 \leq a \geq b \geq 0$, in which case

$$\begin{cases} \text{ls}(\Delta^{(1)}) = \text{ls}(\Delta) - (a + 3) & \text{if } a = b, \\ \text{ls}(\Delta^{(1)}) = \text{ls}(\Delta) - (a + 2) & \text{if } a > b; \end{cases}$$

- $\Delta \cong 2\Sigma$, $\Delta \cong \text{conv}\{(0,0), (4,0), (0,2)\}$, or $\Delta \cong \square_{a,b}$ for certain $a, b \geq 2$, in which case $\text{ls}(\Delta^{(1)}) = \text{ls}(\Delta) - 4$;
- there exist parallel edges $\tau \subset \Delta$ and $\tau' \subset \Delta^{(1)}$ whose supporting lines are at integral distance 1 of each other, such that

$$\#(\tau \cap \mathbb{Z}^2) \geq \#(\tau' \cap \mathbb{Z}^2) + 4;$$

in this case $\text{ls}(\Delta^{(1)}) = \text{ls}(\Delta) - (\#(\tau \cap \mathbb{Z}^2) - \#(\tau' \cap \mathbb{Z}^2))$.

PROOF. For the Lawrence prisms, 2Σ , and $\text{conv}\{(0,0), (4,0), (0,2)\}$, the statement about $\text{ls}(\Delta^{(1)})$ is immediate, while the polygons $\square_{a,b}$ are covered by Lemma 3 and the observation that $(\square_{a,b})^{(1)} \cong \square_{a-2,b-2}$. The last statement follows from Lemma 4.

By (2) it remains to show that in all other situations $\text{ls}(\Delta^{(1)}) \geq \text{ls}(\Delta) - 3$. The cases where $\Delta^{(1)}$ is not two-dimensional can be analyzed explicitly using Koelman's classification: see [1, Thm. 10] or [6, Ch. 4]. We can therefore assume that $\Delta^{(1)}$ is two-dimensional. Let $s = \text{ls}(\Delta^{(1)})$, so that we can suppose that $\Delta^{(1)} \subset s\Sigma$. If

$$\Delta^{(1)(-1)} \subset (s\Sigma)^{(-1)} \tag{3}$$

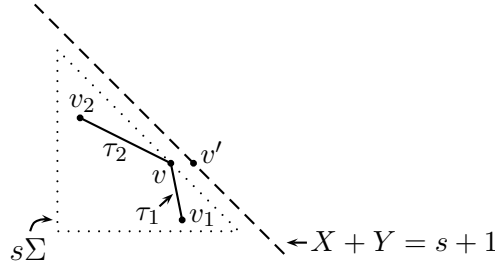
then the theorem follows because $\Delta \subset \Delta^{(1)(-1)}$ and $(s\Sigma)^{(-1)} \cong (s+3)\Sigma$. So let us assume that (3) is not satisfied. Without loss of generality we may then suppose that $\Delta^{(1)(-1)}$ is not contained in the half-plane

$$X + Y \leq s + 1.$$

This means that the edge of $s\Sigma$ connecting $(s, 0)$ and $(0, s)$ cannot contain two vertices of $\Delta^{(1)}$. But it must contain at least one vertex v of $\Delta^{(1)}$: if not, $\Delta^{(1)}$ would be contained in $(s-1)\Sigma$, contradicting $s = \text{ls}(\Delta^{(1)})$.

Write $v = (a, s-a)$ for some $a \in \{0, \dots, s\}$. We distinguish between two cases.

- Assume that v lies in the interior of the edge of $s\Sigma$ that connects $(s, 0)$ and $(0, s)$, i.e. $a \notin \{0, s\}$. Let $v_1 = (a_1, b_1)$ and $v_2 = (a_2, b_2)$ be the vertices of $\Delta^{(1)}$ that are adjacent to v , ordered counterclockwise, and for $i = 1, 2$ let τ_i be the edge connecting v_i and v . Note that $b_1 < s-a$: otherwise $\Delta^{(1)}$ would be contained in $\text{conv}\{(0, s-a), (a, s-a), (0, s)\} \cong a\Sigma$, which would contradict $s = \text{ls}(\Delta^{(1)})$. This means that the outward shift $\tau_1^{(-1)}$ must intersect the line segment spanned by $v = (a, s-a)$ and $v' = (a+1, s-a)$.



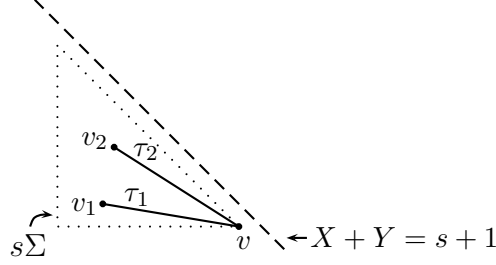
But then $b_2 \leq s-a$, otherwise $\tau_2^{(-1)}$ would also pass in between v and v' , implying that $\tau_1^{(-1)}$ and $\tau_2^{(-1)}$ intersect in the half-plane $X + Y \leq s + 1$: a contradiction. We conclude that $\Delta^{(1)}$ must be contained below the line $Y = s - a$. By symmetry of arguments, it must also lie to the left of $X = a$. Thus $\Delta^{(1)}$ is contained in the rectangle

$$\text{conv}\{(0, 0), (a, 0), (a, s-a), (0, s-a)\}.$$

Now if any of these four vertices would not appear as an actual vertex of $\Delta^{(1)}$ then we would again contradict $s = \text{ls}(\Delta^{(1)})$. Thus $\Delta^{(1)}$ must be exactly this rectangle, and $\Delta^{(1)(-1)} \cong \square_{a+2, s-a+2}$. The case $\Delta = \Delta^{(1)(-1)}$ being among our exceptions, we can assume that at least one of the four vertices of $\Delta^{(1)(-1)}$ does not appear as an actual vertex of Δ . But then $\text{ls}(\Delta) \leq s + 3$, as desired.

- Assume that v is an endpoint of the edge of $s\Sigma$ connecting $(s, 0)$ and $(0, s)$, i.e. $a \in \{0, s\}$. Without loss of generality we may assume that $a = 0$. Again

let $v_1 = (a_1, b_1)$ and $v_2 = (a_2, b_2)$ be the vertices of $\Delta^{(1)}$ that are adjacent to v , ordered counterclockwise, and for $i = 1, 2$ let τ_i be the edge connecting v_i and v .

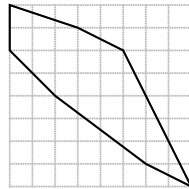


We claim that $v_1 = (0, 0)$, i.e. $a_1 = b_1 = 0$. Indeed:

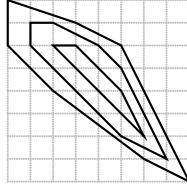
- Assume that $b_1 = 0$. Then $\tau_1^{(-1)}$ is the line $Y = -1$. Since $\tau_2^{(-1)}$ must intersect this line in a lattice point outside the half-plane $X + Y \leq s + 1$ we find (as in the proof of Lemma 5) that τ_2 has slope at most $1/2$ (in absolute value), i.e. $a_2 \leq s - 2b_2$. From this it follows that $a_1 = 0$: if not, the unimodular transformation $(i, j) \mapsto (i + j - 1, j)$ maps $\Delta^{(1)}$ inside $(s - 1)\Sigma$, contradicting $s = \text{ls}(\Delta^{(1)})$.
- Assume that $b_1 \neq 0$. If $a_2 \leq s - 2b_2$ then we would again find a contradiction with $s = \text{ls}(\Delta^{(1)})$. Therefore $a_2 > s - 2b_2$, and by symmetry of arguments also $a_1 < s - 2b_1$. But then $\tau_1^{(-1)}$ passes through or above the point $(s + 2, -1)$, while $\tau_2^{(-1)}$ passes through or to the left of $(s + 2, -1)$. Taking into account their respective slopes, one sees that these lines must intersect in the half-plane $X + Y \leq s + 1$: a contradiction. So this case cannot occur.

Thus $\tau_1 = \text{conv}\{(0, 0), (s, 0)\}$. Now consider the face $\tau = \tau_1^{(-1)} \cap \Delta$ of Δ . The case $\sharp(\tau \cap \mathbb{Z}^2) \geq s + 4$ being among our exceptions, we can assume that $\sharp(\tau \cap \mathbb{Z}^2) \leq s + 3$. The theorem then follows from Lemma 5. ■

Theorem 6 gives a recursive method for computing the lattice size in practice. For example, let Δ be the lattice polygon below.



By taking consecutive interiors, we find the following ‘onion skins’.



The inner polygon is (equivalent to) a Lawrence prism with $a = 4$ and $b = 2$, while the subsequent steps are not exceptional. We find $\text{ls}(\Delta) = \text{ls}(\emptyset) + 6 + 3 + 3 = 10$.

A Magma implementation of this method can be found in the file `basic_commands.m` that accompanies [2].

4 Proof of the main theorem

Let

$$f = \sum_{(i,j) \in \mathbb{Z}^2} c_{i,j}(x,y)^{(i,j)} \in k[x^{\pm 1}, y^{\pm 1}]$$

(where $(x,y)^{(i,j)}$ just means $x^i y^j$) be an irreducible Laurent polynomial with Newton polygon $\Delta(f)$. If $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a unimodular transformation, then the Laurent polynomial

$$f^\varphi = \sum_{(i,j) \in \mathbb{Z}^2} c_{i,j}(x,y)^{\varphi(i,j)}$$

satisfies $\Delta(f^\varphi) = \varphi(\Delta(f))$. Since $U_f \cong U_{f^\varphi}$, it follows that U_f has a plane model, the Newton polygon of whose defining polynomial is contained in $\text{ls}(\Delta(f))\Sigma$. Therefore $s_2(U_f) \leq \text{ls}(\Delta(f))$.

The proof of Theorem 1 is a refinement of this argument.

PROOF OF THEOREM 1. First remark that inequality (1) is trivial as soon as U_f is a rational curve, because the right-hand side is always at least 1. In particular, if $\Delta(f)$ is not two-dimensional (in which case it is necessarily a line segment of integral length 1, by the irreducibility of f) then there is nothing to prove. So we can assume that $\Delta(f)$ is two-dimensional. Furthermore, $\Delta(f)$ should be among the exceptional polygons listed in Theorem 6; if not, the theorem follows from $\text{ls}(\Delta(f)^{(1)}) = \text{ls}(\Delta(f)) - 3$ along with $s_2(U_f) \leq \text{ls}(\Delta(f))$.

If $\Delta(f)$ is a Lawrence prism or 2Σ , then again U_f is rational and there is nothing to prove.

If $\Delta(f) = \text{conv}\{(0,0), (4,0), (0,2)\}$ then U_f is rational or elliptic, so $s_2(U_f) \leq 3 = \text{ls}(\Delta(f)^{(1)}) + 3$.

Assume that $\Delta(f) = \square_{a,b}$ for certain $a, b \geq 2$, and let $(x_0, y_0) \in U_f$. Then the Newton polygon of $f(x + x_0, y + y_0)$ is contained in

$$\text{conv}\{(1, 0), (a, 0), (a, b), (0, b), (0, 1)\}.$$

But then $x^a y^b f(x^{-1} + x_0, y^{-1} + y_0)$ is a polynomial of degree at most $a + b - 1$. So $s_2(U_f) \leq \text{ls}(\Delta(f)) - 1 = \text{ls}(\Delta(f)^{(1)}) + 3$.

Finally, assume that there exist parallel edges $\tau \subset \Delta(f)$ and $\tau' \subset \Delta(f)^{(1)}$ whose supporting lines are at integral distance 1 of each other, of respective lengths r and s , such that $r \geq s + 4$. Similar to the proof of Lemma 4 we can assume that $\tau = \text{conv}\{(0, 0), (r, 0)\}$ and $\tau' = \text{conv}\{(1, 1), (s + 1, 1)\}$. This configuration implies that $\Delta(f)$ is contained in the half-planes $X \geq 0$, $Y \geq 0$ and $X + (r - s - 2)Y \leq r$. In other words,

$$f = \sum_{j=0}^{\lfloor r/(r-s-2) \rfloor} g_j(x) y^j$$

for polynomials $g_j \in k[x]$ satisfying $\deg g_j \leq r - (r - s - 2)j$ and $\deg g_0 = r$. Now factor $g_0(x) = g'_0(x) h'_0(x)$ with $\deg g'_0 = s + 3$ and $\deg h'_0 = r - s - 3$, substitute $y \leftarrow y h'_0(x)$, and kill a factor $h'_0(x)$ to obtain

$$g'_0(x) + \sum_{j=1}^{\lfloor r/(r-s-2) \rfloor} g_j(x) h'_0(x)^{j-1} y^j.$$

One verifies that each term has degree at most $s + 3$, which proves that $s_2(U_f) \leq s + 3 = \text{ls}(\Delta(f)^{(1)}) + 3$. This proves inequality (1).

It remains to verify the bounds in the case where $\Delta(f) \cong d\Upsilon$ for some $d \geq 2$. Note that by Theorem 6 we have $\text{ls}(d\Upsilon) = 3d$, so the bound we need to prove is sharper. Consider the embedding

$$\psi : \mathbb{T}^2 \hookrightarrow \mathbb{P}^3 = \text{Proj}[X_0, X_1, X_2, X_3] : (x, y) \mapsto (x^{-1}y^{-1} : x : y : 1).$$

It embeds U_f in a projective curve C_f which arises as the intersection of the cubic $X_0 X_1 X_2 - X_3^3$ and an irreducible hypersurface of degree d , whose concrete equation depends on f . The hyperplane sections then form a g_{3d}^3 on C_f . By [4, IV.Prop. 3.8 and IV.Thm. 3.9] we can find a point on C_f , the general secant line through which is not a multisecant. Projecting from such a point yields a birational equivalence between C_f and a plane curve of degree $3d - 1$, as wanted. \blacksquare

5 Cases where the bound is sharp

In this section we make extensive reference to a subsequent, more elaborate paper [2] on linear pencils that are encoded in the Newton polygon, in the study of which

the lattice size pops up as a convenient notion [2, Thm. 7.2]. No statement in [2] makes use of any of the results of this section.

We will show that the bound from Theorem 1 often gives the correct value of $s_2(U_f)$. In fact, our guess would be that this is generically true. That is, let Δ be a two-dimensional lattice polygon. Then we conjecture that the set of Laurent polynomials $f \in k[x^{\pm 1}, y^{\pm 1}]$ for which $s_2(U_f)$ meets the (sharpest applicable) bound from Theorem 1 lies Zariski dense in the space of Laurent polynomials that are supported on Δ . More precisely, toric geometry provides a way of associating to Δ a completion of \mathbb{T}^2 , which we denote by $\text{Tor}(\Delta)$. Write C_f for the completion of U_f inside $\text{Tor}(\Delta(f))$. Then:

Conjecture 7. *If C_f is a non-singular projective curve then $s_2(U_f) = \text{ls}(\Delta(f)^{(1)}) + 3$, unless $\Delta(f) \cong d\Upsilon$ for some $d \geq 2$, in which case $s_2(U_f) = 3d - 1$.*

We refer to [2] and the references therein for more background on the construction of $\text{Tor}(\Delta)$ and C_f , and on why the non-singularity of the latter is indeed generically satisfied. We also refer to [2] and its references for similar (provable) interpretations for various other invariants, such as the geometric genus (which equals $\#(\Delta(f)^{(1)} \cap \mathbb{Z}^2)$) and the gonality (which equals $\text{lw}(\Delta(f)^{(1)}) + 2$, except if $\Delta(f) \cong 2\Upsilon$ in which case it is 3).

Note that by Lemma 2 we have $-1 \leq \text{lw}(\Delta(f)^{(1)}) \leq \text{ls}(\Delta(f)^{(1)})$ as soon as $\Delta(f)^{(1)} \neq \emptyset$. We can prove Conjecture 7 near both ends of this spectrum.

Theorem 8. *If $\text{lw}(\Delta(f)^{(1)}) \leq 1$ or $\text{lw}(\Delta(f)^{(1)}) \geq \text{ls}(\Delta(f)^{(1)}) - 2$ then Conjecture 7 is true.*

PROOF. If $\text{lw}(\Delta(f)^{(1)}) = -1$ then U_f is rational and there is nothing to prove.

If $\text{lw}(\Delta(f)^{(1)}) = 0$, then $\Delta(f)^{(1)}$ is a line segment of integral length $g - 1$, with g the geometric genus of $U(f)$, and U_f is hyperelliptic. So $s_2(U_f) = g + 2$, which indeed equals $\text{ls}(\Delta(f)^{(1)}) + 3$.

If $\text{lw}(\Delta(f)^{(1)}) = 1$ then $\Delta(f)^{(1)}$ is equivalent to a Lawrence prism

$$\begin{array}{ccc} (0, 1) & & (b, 1) \\ \hline (0, 0) & & (a, 0) \end{array}$$

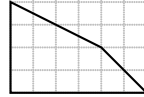
with $1 \leq a \geq b \geq 0$. In this case C_f is a trigonal curve with scrollar invariants a, b by [2, Thm. 9.1]. From [5, Lem. 2.1], which is expressed in terms of the single Maroni invariant $a = \max(a, b)$, we conclude that $s_2(U_f) = a + 3$ if $a > b$, and $s_2(U_f) = a + 4$ if $a = b$. By Theorem 6, in both cases this exactly matches with $\text{ls}(\Delta(f)^{(1)}) + 3$.

If $\text{lw}(\Delta(f)^{(1)}) = \text{ls}(\Delta(f)^{(1)}) - 2$ then by [2, Thm. 7.2] one of the following statements is true:

- $\Delta(f)^{(1)} \cong 2\Upsilon$. In this case C_f is a curve of gonality 6 and geometric genus 10. If C_f would be birationally equivalent to a plane curve of degree $d \leq 7$, by genus considerations we necessarily have $d = 6$ or $d = 7$. But both cases contradict that the gonality is 6, either by projection from any point on this curve (in the $d = 6$ case) or by projection from a singular point on this curve (in the $d = 7$ case, where such a singular point must exist, again by genus considerations). Therefore $s_2(U_f) \geq 8$.
- $\Delta(f)^{(1)} \cong \text{conv}\{(0, 0), (1, 0), (3, 1), (2, 2), (1, 2)\}$, so that $\Delta(f)$ is of the form



(the dashed line indicates $\Delta(f)^{(1)}$). Here C_f is a curve of gonality 4 and geometric genus 7. By [2, Thm. 6.1] and [2, Thm. 9.1] the gonality pencil is unique and its scrollar invariants are 1, 1, and 2. Now suppose that C_f is birationally equivalent to a plane curve of degree $d \leq 6$. By genus considerations we necessarily have $d = 6$. Because the gonality is 4 and the gonality pencil is unique, the curve must have a unique singular point P , of multiplicity 2. Genus considerations yield that P is neither a node nor an ordinary cusp. In particular, the curve has a unique tangent line at P , which intersects the curve at P with multiplicity at least 4. By positioning P at $(0 : 1 : 0)$ in such a way that this tangent line becomes the line at infinity, we see that C_f is birationally equivalent to a plane affine curve of degree 6, whose defining polynomial $g(x', y')$ is supported on the polygon below.



Note that at least one of the coefficients at $x'^4 y'^2$, $x'^5 y'$, x'^6 is non-zero (because the degree is 6) and that the coefficient at y'^4 is non-zero as well (because the gonality is 4). In particular, our g_4^1 is given by the projection map $(x', y') \mapsto x'$. Now let $D_{x'} \in g_4^1$ be the zero divisor of x'^{-1} , and similarly let $D_{y'}$ be the zero divisor of y'^{-1} . The steepness of the above polygon ensures that $D_{y'} \leq 2D_{x'}$. In particular $H^0(2D_{x'}) \supset \{1, x'^{-1}, x'^{-2}, y'^{-1}\}$ is at least 4-dimensional. This shows that 0 must be among the scrollar invariants of our g_4^1 : a contradiction. We conclude that $s_2(U_f) \geq 7$.

- The number of base-point free $g_{\gamma+1}^1$'s on C_f is finite, where γ is the gonality of C_f . It follows that $s_2(U_f) \geq \gamma + 3$, because $s_2(U_f) = \gamma + 2$ resp. $s_2(U_f) = \gamma + 1$

would yield infinitely many base-point free $g_{\gamma+1}^1$'s, obtained by projecting from a non-singular point on the degree $\gamma+2$ model resp. a point outside the degree $\gamma+1$ model, while $s_2(U_f) \leq \gamma$ would contradict that γ is the gonality. But then

$$s_2(U_f) \geq \gamma + 3 = \text{lw}(\Delta(f)^{(1)}) + 5 = \text{ls}(\Delta(f)^{(1)}) + 3,$$

where the first equality follows from [2, Cor. 6.2].

So in each case, the upper bound from Theorem 1 is met.

If $\text{lw}(\Delta(f)^{(1)}) = \text{ls}(\Delta(f)^{(1)}) - 1$ then by [2, Cor. 6.3] we similarly have that one of the following statements holds:

- $\Delta(f)^{(1)} \cong \Upsilon$. In this case C_f is a curve of genus 4, and by genus considerations we find $s_2(U_f) \geq 5$.
- $\Delta(f)^{(1)}$ is equivalent to one of
 - $\text{conv}\{(1, 0), (2, 1), (1, 2), (0, 1)\}$,
 - $\text{conv}\{(1, 0), (2, 0), (1, 2), (0, 1)\}$,
 - $\text{conv}\{(0, 0), (2, 0), (1, 2)\}$,

so that $\Delta(f)$ is among the following polygons.



In either case C_f is a curve of genus 5 and gonality 4. If C_f would be birationally equivalent to a plane curve of degree strictly less than 6, then by genus considerations it necessarily concerns a singular curve of degree 5. Projecting from a singular point contradicts that the gonality is 4. So $s_2(U_f) \geq 6$.

- $\Delta(f)^{(1)} \not\cong \Upsilon$ and the number of g_γ^1 's on C_f is finite, where γ is the gonality. In this case, as above we conclude that $s_2(U_f) \geq \gamma + 2$ and hence

$$s_2(U_f) \geq \gamma + 2 = \text{lw}(\Delta(f)^{(1)}) + 4 = \text{ls}(\Delta(f)^{(1)}) + 3,$$

where the first equality again follows from [2, Cor. 6.2] (here we used that $\Delta(f)^{(1)} \not\cong \Upsilon$).

Again, in each of these cases, the upper bound from Theorem 1 is met.

Finally, if $\text{lw}(\Delta(f)^{(1)}) = \text{ls}(\Delta(f)^{(1)})$ then by Lemma 2 we have that $\Delta(f)^{(1)} \cong (d-3)\Sigma$ for some integer $d \geq 3$. Then C_f is a smooth plane curve of degree d by [2, Cor. 12], and therefore $s_2(U_f) = d = \text{ls}(\Delta(f)^{(1)}) + 3$. ■

Corollary 9. *Conjecture 7 is true if $\sharp(\Delta(f)^{(1)} \cap \mathbb{Z}^2) \leq 7$, i.e., if the genus of C_f is ≤ 7 .*

PROOF. All interior lattice polygons satisfying this bound can be easily listed (up to equivalence), either by hand or using the data from [1]. All of these turn out to be covered by the previous lemma. ■

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